Rayleigh Waves Generated by a Thermal Source: A Three-Dimensional Transient Thermoelasticity Solution

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A three-dimensional transient thermoelastic solution is obtained for Rayleigh-type disturbances propagating on the surface of a half-space. These surface waves are generated by either a buried or surface thermal source, which has the form of a concentrated heat flux applied impulsively. In an effort to model this problem as realistically as possible, the half-space material is taken to respond according to Biot's fully coupled thermoelasticity. The problem has relevance to situations involving heat generation due to: (i) laser action (impulsive electromagnetic radiation) on a surface target, (ii) underground nuclear activity, and (iii) friction developed during underground fault motions related to seismic activity. The problem was attacked with unilateral and double bilateral Laplace transforms, which suppress, respectively, the time variable and two of the space variables. The Rayleigh wave contribution is obtained as a closed-form expression by utilizing asymptotics, complex-variable theory and certain results for Bessel functions. The dependence of the normal displacement associated with the Rayleigh wave upon the distance from the source epicenter and the distance from the wavefront is also determined.
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1 Introduction

A class of interesting problems of thermomechanical wave motions arises from the action of a thermal source in a conducting and deformable body. The source can be situated either on the surface or inside the medium (buried source). Typical problems of this class involve: (i) laser action (impulsive electromagnetic radiation) on a surface target (see e.g. Morland [1], Sve and Miklowitz [2], Bechtel [3], Hetnarski and Ignaczak [4], and Royer and Chenu [5]), (ii) underground nuclear activity (see e.g., Bullen and Bolt [6]), and (iii) friction developed during underground fault motions related to seismic activity (see e.g., Kanamori et al. [7]). In many cases, these problems can be viewed as a three-dimensional (3D) situation involving a thermoelastic half-space under either a surface or buried heat source. This situation is studied here by employing the coupled inertial thermoelasticity theory of Biot [8] (see also Achenbach [9]). In particular, we focus attention on the surface disturbance of the Rayleigh-type and provide a closed-form expression for the associated displacement field. Indeed, past experience with pure mechanical (i.e., without any thermal effects) versions of the present problem indicates that the Rayleigh-wave disturbance is the *dominant* one over the surface after a certain time (see e.g., the 2D analysis of Garvin [10] involving a buried dilatational source in a half-plane and the 3D analysis of Pekeris and Lifson [11] involving a buried concentrated vertical force in a half-space).

We should mention that a recent study by the present authors and Brock (Lykotrafitis, Georgiadis, and Brock [12]) dealt with the same problem studied here and provided an evaluation of the complete field at the surface. This field comprises thermoelastic dilatational and Rayleigh waves, and elastic shear waves. However, the latter study relies much upon numerical analysis (involving numerical wavenumber integrations and numerical Laplace-transform inversions) and does not furnish an analytical expression for the evaluation of the surface displacements. Instead, our aim here is to provide a simple *closed-form* expression for the Rayleigh-wave disturbance without using any special numerical technique. This was made possible by using asymptotics and certain results of complex-variable theory and Bessel functions in addition to the basic integral-transform analysis of Ref. [12]. The key idea used is making explicit the appearance of Rayleigh-wave poles by obtaining an approximate form of the Rayleigh function that exhibits no dispersion but still depends upon the thermoelastic constants. Notably, this approximate form is numerically very close to the exact one giving therefore very accurate results.

It should be mentioned that most of the studies published before on wave propagation induced by sudden heating model the problems as *one-dimensional* (see e.g., Boley and Tolins [13], and Hetnarski and Ignaczak [14]), employ *uncoupled* thermoelasticity (see e.g., Sve and Miklowitz [2]) or treat only *infinite* domains, i.e., full spaces (see e.g., Predeleanu [15], Fleurier and Predeleanu [16], Sharp and Crouch [17], and Manolis and Beskos [18,19]). Also, some of the aforementioned works consider the special case of a *time-harmonic* response. On the contrary, the present study aims at a more realistic formulation of these problems and is therefore based on the transient coupled inertial thermoelasticity, while it treats a three-dimensional problem in a half-space domain. Notice that the relevance of the constitutive theory used here to *thermal-shock* problems—particularly the importance of inertial and thermal-coupling effects—was shown in the studies of Hetnarski [20], Boley and Tolins [13], Sternberg and Chakravorty [21,22], and Francis [23]. More recent work employing this theory in transient problems of wave propagation and fracture was

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done by, among others, Atkinson and Craster [24], Brock [25], Brock, Rodgers and Georgiadis [26], Brock and Georgiadis [27], Georgiadis, Brock, and Rigatos [28], and Georgiadis, Rigatos, and Brock [29]. Finally, within the context of a different theory, namely inertialess and uncoupled thermoelasticity, solutions for thermally activated surface displacements in a half-space were obtained by Barber [30] and Barber and Martin-Moran [31].

2 Problem Statement

Consider a 3D body in the form of a half-space $z>-H$ (see Fig. 1) which is both thermally conducting and deformable. The body is initially at rest and at uniform temperature $\tilde{T}_0$. At time $t=0$, a thermal source acts at a point situated at a depth H below the surface. This point of the half-space is taken as the origin of the Cartesian coordinate system (x,y,z). A concentrated thermal source having an impulsive time variation is assumed, with the understanding that the solution of this problem (Green's function or fundamental solution) can be integrated in space and time to give then the solution for any general thermal loading. Also, the source has an intensity KQ, where K is the thermal conductivity with dimensions of (power)(length)$^{-1}$(°C)$^{-1}$, °C means degrees of temperature and Q is a multiplier expressed in (°C)(length)(time).

Then, according to the linear, isotropic, inertial coupled thermoelasticity theory (Biot [8], Achenbach [9], Chadwick [32], and Carlson [33]), the governing equations for this problem are written as

$$\boldsymbol{\sigma}=\mu(\nabla\mathbf{u}+\mathbf{u}\nabla)+\lambda(\nabla\cdot\mathbf{u})\mathbf{1}-\kappa_0(3\lambda+2\mu)\theta\mathbf{1}, \tag{1}$$

$$\mathbf{q}=-K\nabla\theta, \tag{2}$$

$$\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})-\kappa_0(3\lambda+2\mu)\nabla\theta=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{3}$$

$$K\nabla^2\theta-\rho c_v\frac{\partial\theta}{\partial t}-\kappa_0(3\lambda+2\mu)\tilde{T}_0\frac{\partial(\nabla\cdot\mathbf{u})}{\partial t} +KQ\cdot\delta(t)\cdot\delta(x)\cdot\delta(y)\cdot\delta(z)=0, \tag{4}$$

where (1) is the Neumann-Duhamel law, (2) is the heat-conduction Fourier law, (3) is the displacement-temperature equation of motion, and (4) is the coupled heat equation. Also, in the above equations, $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{u}$ is the displacement vector, $\theta=\tilde{T}-\tilde{T}_0$ is the change in temperature, $\tilde{T}$ is the current temperature, $\tilde{T}_0$ is the initial temperature, $\mathbf{q}$ is the heat-flux vector, (λ,μ) are the Lame constants, κ_0 is the coefficient of linear expansion, ρ is the mass density, c_v is the specific heat at constant deformation, $\delta(\;)$ is the Dirac delta distribution, $\mathbf{1}$ is the identity tensor, ∇ is the gradient operator, and $\nabla^2=(\partial^2/\partial x^2)+(\partial^2/\partial y^2)+(\partial^2/\partial z^2)$ is the Laplace operator. All field quantities above are functions of (x,y,z,t).

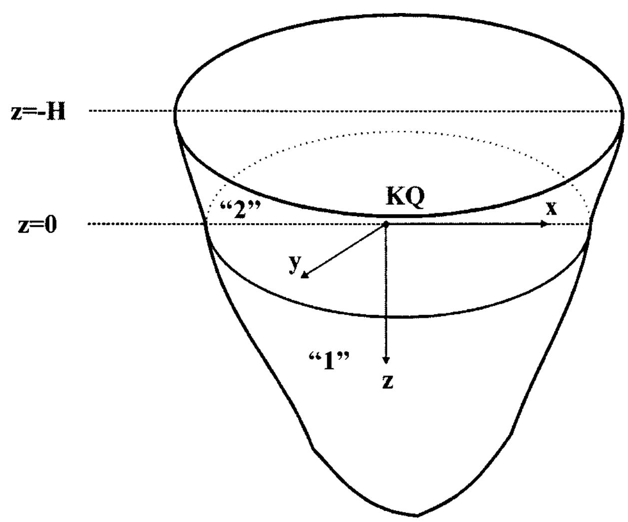

Fig. 1 A thermally conducting and deformable body in the form of 3D half-space under the action of a buried ($H\neq0$) or surface ($H=0$) heat source

In addition, zero initial conditions are taken, i.e.

$$\mathbf{u}=\partial\mathbf{u}/\partial t=\theta=0 \quad \text{for } t\leqslant0 \text{ in } (-\infty<x<\infty,-\infty<y<\infty, -H<z<\infty), \tag{5}$$

and we also assume that the half-space surface $z=-H$ is traction free and insulated (i.e., no heat is conducted through the half-space surface and air). Finally, the pertinent *finiteness* conditions at remote regions (Ignaczak and Nowacki [34]) state that the field at infinity remains bounded although temperature signals travel—according to Biot's theory—at an infinite speed.

The objective of the present work is to determine the vertical displacement at the surface for the problem described by Eqs. (1)–(5). The solution of this problem is greatly facilitated by removing the source term in (4) and considering this term as a discontinuity along an *imagined plane* at $z=0$. This strategy was introduced first by Pekeris [35] (see also Miklowitz [36]) in treating the pure mechanical problem of a half-space under a buried vertical force. Considering thus an imaginary plane along $z=0$ that separates the original half-space into the half-space $0<z<\infty$ (region 1 in Fig. 1) and the strip $-H<z<0$ (region 2 in Fig. 1), we write the pertinent continuity and discontinuity conditions at $z=0$ along with the standard boundary conditions at $z=-H$

$$\mathbf{u}^{(1)}(x,y,0,t)=\mathbf{u}^{(2)}(x,y,0,t), \tag{6a}$$

$$\theta^{(1)}(x,y,0,t)=\theta^{(2)}(x,y,0,t), \tag{6b}$$

$$\sigma_{zj}^{(1)}(x,y,0,t)=\sigma_{zj}^{(2)}(x,y,0,t), \tag{6c}$$

$$\frac{\partial\theta^{(1)}(x,y,0,t)}{\partial z}-\frac{\partial\theta^{(2)}(x,y,0,t)}{\partial z}=Q\cdot\delta(t)\cdot\delta(x)\cdot\delta(y), \tag{6d}$$

$$\sigma_{zj}(x,y,-H,t)=0, \tag{7a}$$

$$\frac{\partial\theta(x,y,-H,t)}{\partial z}=0, \tag{7b}$$

where $-\infty<x<\infty$, $-\infty<y<\infty$, $(j=x,y,z)$, and the superscript in parentheses 1 or 2 attached to a field quantity means that the plane $z=0$ is approached as $z\to0^+$ or $z\to0^-$, respectively.

In this way, the original problem (1)–(5) and (7) is equivalent to the problem described by (1)–(3) and (5)–(7) and with the equation $K\nabla^2\theta-\rho c_v(\partial\theta/\partial t)-\kappa_0(3\lambda+2\mu)\tilde{T}_0(\partial(\nabla\cdot\mathbf{u})/\partial t)=0$ replacing now Eq. (4). Further, a convenient normalization is performed allowing the two field equations of the problem [i.e., Eqs. (3) and (4) with no source terms] to take the form

$$\nabla^2\mathbf{u}+(m^2-1)\nabla\Delta+\kappa\nabla\theta-m^2\frac{\partial^2\mathbf{u}}{\partial s^2}=0, \tag{8}$$

$$\frac{\kappa}{m^2}\nabla^2\theta-\frac{\kappa}{hm^2}\frac{\partial\theta}{\partial s}+\frac{\varepsilon}{h}\frac{\partial\Delta}{\partial s}=0, \tag{9}$$

where $s=V_1t$ is the normalized time (with dimension of length),-

$V_1=[(\lambda+2\mu)/\rho]^{1/2}$ is the dilatational-wave velocity in the *absence* of thermal effects (i.e., within the "pure" mechanical theory), $\kappa=-\kappa_0(3\lambda+2\mu)/\mu=\kappa_0(4-3m^2)<0$ is the normalized coefficient of linear expansion, $\varepsilon=(\tilde{T}_0/c_v)(\kappa V_2/m)^2$ is the dimensionless coupling coefficient, $h=(KV_2/\mu m c_v)$ is the thermoelastic characteristic length, $V_2=(\mu/\rho)^{1/2}$ is the shear-wave velocity, $m=V_1/V_2>1$, and $\Delta=\nabla\cdot\mathbf{u}$ is the dilatation. As regards the range of numerical values that ε and h take on, for most materials the characteristic length is very small [typically $h=O(10^{-10}\,\mathrm{m})$, see, e.g., Chadwick [32]] but the coupling coefficient can be as high as $\varepsilon=O(10^{-1})$ (e.g., $\varepsilon=0.36$ for Polycarbonate at $\tilde{T}_0=40°\mathrm{C}$). The fact that h is very small with respect to s for a rather wide time-range will be conveniently utilized in the ensuing analysis.

3 Basic Integral-Transform Analysis

This section essentially reproduces relevant material from our recent related work (Lykotrafitis, Georgiadis, and Brock [12]). This material is briefly presented here for the sake of completeness and because of the need to introduce certain definitions. It is also emphasized that although the form of conditions (6) and (7) suggest existence of an axisymmetric field, the basic integral-transform analysis presented here is appropriate for more general *nonaxisymmetric* situations. This is why we do not use the Hankel transform below. Certainly, the fact that we deal with an axisymmetric field in our specific problem will emerge in the course of solving the problem.

The dependence of the problem on the variables (x,y,s) is suppressed through the use of multiple Laplace transforms (see e.g., van der Pol and Bremmer [37], and Carrier et al. [38]). The unilateral transform pair (direct and inverse transform) is defined as

$$\Phi(x,y,z,p)=\int_0^\infty \varphi(x,y,z,s)\cdot e^{-ps}ds, \tag{10a}$$

$$\varphi(x,y,z,s)=(1/2\pi \mathrm{i})\int_{\Gamma_1}\Phi(x,y,z,p)\cdot e^{ps}dp, \tag{10b}$$

and the direct transform suppresses the timelike variable s. The double bilateral transform pair is defined as

$$\Phi^*(q,w,z,p)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(x,y,z,p)\cdot e^{-p(qx+wy)}dxdy, \tag{11a}$$

$$\Phi(x,y,z,p)=(p/2\pi \mathrm{i})^2\int_{\Gamma_2}\int_{\Gamma_3}\Phi^*(q,w,z,p)\cdot e^{p(qx+wy)}dqdw, \tag{11b}$$

and the direct transform suppresses the space variables (x,y). In what follows, we save a capital letter for the unilateral direct transform, whereas the double bilateral direct transform is denoted by an asterisk. It is also noticed that (van der Pol and Bremmer [37]): (1) Because of Lerch's theorem for the uniqueness of unilateral Laplace transforms and because of the existence of Widder's inversion formula for real p, it is sufficient to view $\Phi(x,y,z,p)$ as a function of a *real* variable p over some segment of the real axis in the half-plane of analyticity. Once $\Phi(x,y,z,p)$ is determined as an explicit function of p in the course of solving the transformed differential equations, its definition can be extended to the whole complex p-plane, except for isolated singular points, through analytic continuation. (2) The variables q and w should be treated as *complex*. (3) The integration path Γ_j, with $(j=1,2,3)$, is a line parallel to the imaginary axis in the associated transform plane and lies *within* the region of analyticity.

Applying now (10*a*) and (11*a*) to the governing equations (1), (8), and (9), and considering (5) yields the following general expressions for the transformed temperature change, displacements and stresses (details of this procedure are given in Appendix A of Ref. [12]). These expressions are, of course, different in the regions 1 and 2 of the original half-space.

(a) Region 1 $(0<z<\infty)$:

$$\begin{bmatrix} \dfrac{\kappa}{m^2}\Theta^* \\ pU_x^* \\ pU_y^* \\ pU_z^* \\ \dfrac{1}{\mu}\Sigma_{xy}^* \\ \dfrac{1}{\mu}\Sigma_{xz}^* \\ \dfrac{1}{\mu}\Sigma_{yz}^* \\ \dfrac{1}{\mu}\Sigma_{xx}^* \\ \dfrac{1}{\mu}\Sigma_{yy}^* \\ \dfrac{1}{\mu}\Sigma_{zz}^* \end{bmatrix} = \begin{bmatrix} M_+ & M_- & 0 & 0 & 0 & 0 & 0 & 0 \\ -q & -q & 1 & 0 & 0 & 0 & 0 & 0 \\ -w & -w & 0 & 1 & 0 & 0 & 0 & 0 \\ a_+ & a_- & \dfrac{q}{\beta} & \dfrac{w}{\beta} & 0 & 0 & 0 & 0 \\ -2qw & -2qw & w & q & 0 & 0 & 0 & 0 \\ 2qa_+ & 2qa_- & -\dfrac{T_w}{\beta} & \dfrac{wq}{\beta} & 0 & 0 & 0 & 0 \\ 2wa_+ & 2wa_- & \dfrac{wq}{\beta} & -\dfrac{T_q}{\beta} & 0 & 0 & 0 & 0 \\ T_{w+} & T_{w-} & 2q & 0 & 0 & 0 & 0 & 0 \\ T_{q+} & T_{q-} & 0 & 2w & 0 & 0 & 0 & 0 \\ -T & -T & -2q & -2w & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1e^{-pa_+z} \\ X_2e^{-pa_-z} \\ X_3e^{-p\beta z} \\ X_4e^{-p\beta z} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{12}$$

(b) Region 2 ($-H<z<0$):

$$\begin{bmatrix} \frac{\kappa}{m^2}\Theta^* \\ pU_x^* \\ pU_y^* \\ pU_z^* \\ \frac{1}{\mu}\Sigma_{xy}^* \\ \frac{1}{\mu}\Sigma_{xz}^* \\ \frac{1}{\mu}\Sigma_{yz}^* \\ \frac{1}{\mu}\Sigma_{xx}^* \\ \frac{1}{\mu}\Sigma_{yy}^* \\ \frac{1}{\mu}\Sigma_{zz}^* \end{bmatrix} = \begin{bmatrix} M_+ & M_+ & M_- & M_- & 0 & 0 & 0 & 0 \\ -q & -q & -q & -q & 1 & 0 & 1 & 0 \\ -w & -w & -w & -w & 0 & 1 & 0 & 1 \\ -a_+ & a_+ & -a_- & a_- & \frac{-q}{\beta} & \frac{-w}{\beta} & \frac{q}{\beta} & \frac{w}{\beta} \\ -2qw & -2qw & -2qw & -2qw & w & q & w & q \\ -2qa_+ & 2qa_+ & -2qa_- & 2qa_- & \frac{T_w}{\beta} & \frac{-wq}{\beta} & \frac{-T_w}{\beta} & \frac{wq}{\beta} \\ -2wa_+ & 2wa_+ & -2wa_- & 2wa_- & \frac{-wq}{\beta} & \frac{T_q}{\beta} & \frac{wq}{\beta} & \frac{-T_q}{\beta} \\ T_{w+} & T_{w+} & T_{w-} & T_{w-} & 2q & 0 & 2q & 0 \\ T_{q+} & T_{q+} & T_{q-} & T_{q-} & 0 & 2w & 0 & 2w \\ -T & -T & -T & -T & -2q & -2w & -2q & -2w \end{bmatrix} \begin{bmatrix} X_5 e^{pa_+z} \\ X_6 e^{-pa_+z} \\ X_7 e^{pa_-z} \\ X_8 e^{-pa_-z} \\ X_9 e^{p\beta z} \\ X_{10} e^{p\beta z} \\ X_{11} e^{-p\beta z} \\ X_{12} e^{-p\beta z} \end{bmatrix} \tag{13}$$

where Θ^* is the multiply-transformed change in temperature, and (U_x^*, U_y^*, U_z^*) and $(\Sigma_{xy}^*, \Sigma_{xz}^*, \ldots, \Sigma_{zz}^*)$ are the multiply-transformed components of, respectively, the displacement vector and the stress tensor. We should also notice that solution (12) is bounded at $z\to\infty$ appropriately satisfying thus the finiteness conditions, whereas such constraints need not be imposed on solution (13). In the above equations, the yet unknown $X_1, X_2, \ldots, X_{12}$ are arbitrary functions of (q,w,p) which have to be determined from the boundary conditions in each specific problem. Also, the following definitions are employed in (12) and (13):

$$a_\pm = (m_\pm^2 - q^2 - w^2)^{1/2}, \tag{14a}$$

$$\beta = (m^2 - q^2 - w^2)^{1/2} \tag{14b}$$

$$m_\pm = \frac{1}{2}\left[\left(1+\frac{1}{(hp)^{1/2}}\right)^2 + \frac{\varepsilon}{hp}\right]^{1/2} \pm \frac{1}{2}\left[\left(1-\frac{1}{(hp)^{1/2}}\right)^2 + \frac{\varepsilon}{hp}\right]^{1/2}, \tag{15}$$

$$M_\pm = m_\pm^2 - 1, \tag{16}$$

$$T = 2\beta^2 - m^2 = m^2 - 2(q^2 + w^2), \tag{17a}$$

$$T_\pm = 2a_\pm^2 - m^2 \tag{17b}$$

$$T_{q\pm} = T_\pm + 2q^2, \tag{18a}$$

$$T_{w\pm} = T_\pm + 2w^2 \tag{18b}$$

$$T_q = T + q^2, \tag{19a}$$

$$T_w = T + w^2. \tag{19b}$$

Further, a new complex variable ζ is defined through $\zeta^2 = q^2 + w^2$ allowing the placement of necessary *branch cuts* for the functions $a_\pm \equiv a_\pm(\zeta, p) = (m_\pm^2 - \zeta^2)^{1/2}$ and $\beta \equiv \beta(\zeta) = (m^2 - \zeta^2)^{1/2}$. These restrictions in the ζ-plane are in accord with the chosen solution forms in (12) and (13). For the representative case of $\beta(\zeta)$, Fig. 2 depicts these branch cuts (the cuts are situated outwards with respect to the origin $\zeta=0$—a similar situation exists for the functions $a_\pm(\zeta,p)$). In this way, it is $\mathrm{Re}\, a_+ \geq 0$, $\mathrm{Re}\, a_- \geq 0$, and $\mathrm{Re}\, \beta \geq 0$ in the cut plane. Also, we record here the two possible arrangements of m_+, m_-, and m with respect to their magnitude. This information in conjunction with the placement of branch cuts will enable the proper definition of the regions of analyticity of various functions appearing in the analysis.

The quantities m_+ and m_- are p-dependent (recall that p is real and non-negative), whereas m is constant. From their definitions, the following inequalities follow:

$$m_- < m_+ < m \quad \text{for } hp > \frac{m^2(1+\varepsilon)-1}{m^2(m^2-1)}, \tag{20a}$$

$$m_- < m < m_+ \quad \text{for } hp < \frac{m^2(1+\varepsilon)-1}{m^2(m^2-1)}. \tag{20b}$$

In addition, useful approximations for the quantities m_+ and m_- can be obtained from (15) by taking $s\to(1/p)$, when s is very small or very large, and by performing series expansion and keeping the dominant terms (see e.g., Carrier et al. [38] for similar procedures). The following approximate forms considerably simplify unilateral Laplace transform inversions

$$m_+ \cong 1 \quad \text{and} \quad m_- \cong \frac{1}{(hp)^{1/2}} \quad \text{for } \frac{s}{h} \ll 1, \tag{21a}$$

$$m_+ \cong \left(\frac{1+\varepsilon}{hp}\right)^{1/2} \quad \text{and} \quad m_- \cong \frac{1}{(1+\varepsilon)^{1/2}} \quad \text{for } \frac{s}{h} \gg 1. \tag{21b}$$

Notice that validity of (20*a*) or (20*b*) is necessary but not sufficient for, respectively, the validity of (21*a*) or (21*b*).

Finally, it turns out that the case in (20*a*) is rather impractical since it corresponds to an *extremely small* initial time interval of the process, which for most conducting materials is $t < O(10^{-13}\,\mathrm{s})$. This is found by taking $s\to(1/p)$ for very small s (i.e., for very small time). In the present study, information is needed generally for longer times so we shall focus interest only on the case (20*b*) and employ (21*b*) appropriately. Any case with, say, $(s/h)\geq 100$ leads to a reasonable approximation for $m_\pm$. The results in (21*b*) are indeed robust because the normalized time is scaled by an extremely small length (the thermoelastic characteristic length).

Now, transforming via (10*a*) and (11*a*) the continuity/discontinuity conditions (6) and the boundary conditions (7), in view also of the general transformed solutions (12) and (13), leads to a linear algebraic system of 12 equations in the 12 unknown

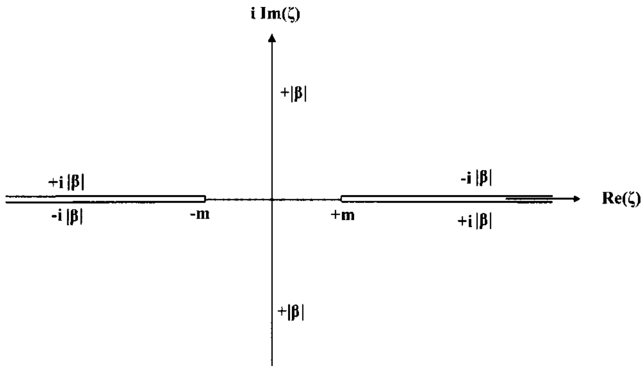

Fig. 2 Branch cuts for the function $\beta(\zeta)\equiv(m^2-\zeta^2)^{1/2}$ in the complex ζ-plane. Similar branch cuts, emanating from the points $m_{\pm}(p)$, are also introduced for the functions $a_{\pm}(\zeta)\equiv(m_{\pm}^2-\zeta^2)^{1/2}$.

$X_1, X_2, \ldots, X_{12}$. Obviously, an exact (i.e., *symbolical* and not numerical) solution to the system is sought here and this was made possible by using MATHEMATICA™. The expressions for $X_1, X_2, \ldots, X_{12}$ are given in Appendix A.

Having available the solution $(X_1, X_2, \ldots, X_{12})$ and therefore, by (12) and (13), the general expressions for the double transformed temperature, displacements and stresses allows determining the field quantities at any point of the original space and at any time instant through successive *inversions* of the type (11*b*) and (10*b*). However, we emphasize at this point that a treatment employing the Cagniard–deHoop technique [9,36,39] to accomplish the transform inversions in an exact manner seems to be impossible due to the very complicated multiple transformed solution in the present problem. In the simpler buried-source problems of non-thermal type such a difficulty was not met and the Cagniard–deHoop technique had successfully been applied (see e.g., Pekeris [35], Garvin [40], and Payton [41]). Indeed, we note that, after the appropriate contour integration involved in the Cagniard–deHoop technique, the integrand in the semi-infinite branch-line integration is still p-dependent and, therefore, the unilateral transform inversion is impossible to be carried out exactly through the standard inspection procedure. For more details on this difficulty, we refer to the work by Georgiadis et al. [29], who treated the counterpart 2D problem and employed an *approximation* at a similar point of the analysis. Their asymptotic approach is, however, different than that employed here (see Sec. 4 below).

We close the presentation of the basic integral-transform analysis by noticing that if, instead of $\delta(t)$, a general dependence from time of the thermal loading in (4) is to be considered (denoted by an arbitrary function $g(t)$), then the quantity Q in the equations of Appendix A has to be replaced by $(Q/V_1)\cdot G(p)$, where $G(p)$ denotes the unilateral Laplace transform of the function $g((s/V_1)\equiv t)$.

4 Transformed Solution and Asymptotic Considerations

In what follows, we focus attention on the evaluation of the vertical displacement at the surface $u_z(x,y,z=-H,t)$. In view of the previous results, the multiply transformed displacement $U_z^*(q,w,z=-H,p)\equiv U_z^*(\zeta,z=-H,p)$ is given by

$$U_z^*(\zeta,z=-H,p)=\kappa Q V_1 \frac{T}{p^2}\frac{a_+e^{-a_-pH}-a_-e^{-a_+pH}}{D(\zeta,p)}, \quad (22)$$

where the functions $a_+(\zeta,p)$ and $a_-(\zeta,p)$, and the complex variable ζ have been defined before. Also, from (17*a*) and the definition of ζ, it is $T=m^2-2\zeta^2$. One may notice that the very definition of the variable ζ and the form of U_z^* in (22) exhibit the *axisymmetric* nature of the problem, a fact that will become evident in the ensuing procedure. Finally, of central importance to the solution for the *surface* disturbances is the function D, which is associated with waves of Rayleigh type. This is given as

$$D\equiv D(\zeta,p)=a_-M_-R_+-a_+M_+R_-, \quad (23)$$

where the functions

$$R_+\equiv R_+(\zeta,p)=4\zeta^2a_+\beta+T^2, \quad (24a)$$

$$R_-\equiv R_-(\zeta,p)=4\zeta^2a_-\beta+T^2, \quad (24b)$$

can be identified as the *thermoelastic* counterparts of the nonthermal pure-elastic Rayleigh function (transformed function), which is given as $R^{\text{elastic}}=4\zeta^2a\beta+T^2$, with $a\equiv a(\zeta)=(1-\zeta^2)^{1/2}$ and $\beta\equiv\beta(\zeta)$ given as before (see e.g., Achenbach [9], Miklowitz [36]). Contrary to the latter case, R_+ and R_- exhibit a p-dependence showing therefore that the thermoelastic Rayleigh waves in the physical space/time domain are dispersive. However, it was shown in the study of Georgiadis, Brock, and Rigatos [28] that generally the thermoelastic Rayleigh-wave velocity varies only slightly with time, a result explained in view of the fact that while there is a strong shear contribution (which remains unaffected by thermal effects) to the Rayleigh waves, the dilatational part of them is very weak (see e.g., Viktorov [42]). We will take advantage of this result immediately initiating the asymptotic considerations to obtain $u_z(x,y,z=-H,t)$.

It will be shown, indeed, that the function D can be expressed in terms of an *approximate* Rayleigh function that exhibits *no dispersion* (i.e., this Rayleigh function does not contain the time transform variable p) but still depends on the coupling constant ε. The approximate form of the function D itself will exhibit dependence upon the thermoelastic constants (ε,h) and the transform variables (ζ,p). First, one may write from (23) and (24) the following expression for the function (D/a_+):

$$\frac{D}{a_+}=4\zeta^2a_-\beta M_+\left(\frac{M_-}{M_+}-1\right)+T^2M_+\left(\frac{a_-M_-}{a_+M_+}-1\right). \quad (25)$$

Now, the terms (M_-/M_+) and $[(a_-M_-)/(a_+M_+)]$ in the above expression, in view of (14*a*) and (16), are written as

$$\frac{M_-}{M_+}=\frac{m_-^2-1}{m_+^2-1} \quad (26a)$$

and

$$\frac{a_-M_-}{a_+M_+}=\frac{(m_-^2-\zeta^2)^{1/2}}{(m_+^2-\zeta^2)^{1/2}}\frac{m_-^2-1}{m_+^2-1}. \quad (26b)$$

Further, when $(s/h)\gg 1$, use of the expressions for m_+ and m_- in either (15) or (21*b*) lead to the results

$$\left|\frac{M_-}{M_+}\right|\ll 1 \quad (27a)$$

and

$$\left|\frac{a_-M_-}{a_+M_+}\right|\ll 1. \quad (27b)$$

To give a numerical estimate, we obtain values of the ratio $[(a_-M_-)/(a_+M_+)]$ for different p's (recall that p is the time Laplace-transform variable). The constants of a model material utilized in the present study to derive numerical results (see Sec. 6 below) are employed. These constants are $\varepsilon=0.011$, $h=1.864\times10^{-9}$ m, and Poisson's ratio $\nu=0.3$ [which gives a ratio of wave velocities $m\equiv(V_1/V_2)=1.8708$]. Also, we take $\zeta=\zeta_R$, which is the value corresponding to the arrival of the Rayleigh wavefront—see Eq. (36) below and which for the model material is calculated to be $\zeta_R=2.0162$. Then, the following values of the ratio in question are obtained:

$$[(a_-M_-)/(a_+M_+)] = -1.73840\times 10^{-11} \quad \text{for } p=10^2,$$

$$[(a_-M_-)/(a_+M_+)] = -1.51502\times 10^{-15} \quad \text{for } p=10^0,$$

$$[(a_-M_-)/(a_+M_+)] = -4.79091\times 10^{-23} \quad \text{for } p=10^{-5},$$

which clearly show that for increasing time (i.e., decreasing p) the ratio rapidly diminishes and can practically be taken equal to zero. The same applies to the ratio (M_-/M_+) as well.

Then, (27) allow writing (25) under the following approximate form:

$$\frac{D}{a_+} = -M_+(4\zeta^2 a_- \beta + T^2), \tag{28}$$

and, since $m_+ \gg 1$ for $(s/h) \gg 1$, one may further obtain in view of (16)

$$\frac{D}{a_+} = -m_+^2(4\zeta^2 a_- \beta + T^2), \tag{29}$$

where in the last two expressions and, also, in what follows the quantities m_+ and m_- assume the forms [taken from (21b)]

$$m_+ = \left(\frac{1+\varepsilon}{hp}\right)^{1/2}, \tag{30a}$$

$$m_- = \frac{1}{(1+\varepsilon)^{1/2}}. \tag{30b}$$

Finally, in view of the above, Eq. (29) becomes

$$\frac{D}{a_+} = -\frac{1+\varepsilon}{hp}\left[(m^2-2\zeta^2)^2 + 4\zeta^2\left(\frac{1}{1+\varepsilon}-\zeta^2\right)^{1/2}(m^2-\zeta^2)^{1/2}\right]$$
$$\equiv -\frac{1+\varepsilon}{hp} R^{\text{therm}}, \tag{31}$$

where the symbol $\equiv$ means equality by definition.

In the above result, the approximate Rayleigh function R^{therm} exhibits no dispersion (i.e., it does not depend upon p) but depends upon the coupling constant ε. All the above approximations will properly be utilized below.

The next step is to determine the zeros of the function (D/a_+), which is given in (31). This information will be utilized later in the inversion procedure. By invoking the principle of the argument (see e.g., Carrier et al. [38], and Ablowitz and Fokas [43]), it can be shown that the two real zeros $\zeta = \pm\zeta_R$ of the function (D/a_+) are the only zeros of this function in the entire ζ-plane. These correspond to axisymmetric thermoelastic Rayleigh wavefronts propagating with a velocity $V_R = V_1/\zeta_R$ along the traction-free half-space surface. Working with real p such that $p>0$ [which, of course, is necessary for the convergence of the integral defining the unilateral Laplace transform in Eq. (10a)] in the case of interest $m_- < m < m_+$ [cf. Eq. (20b)], we can obtain a closed-form expression for the root ζ_R by utilizing factorization operations of the kind encountered in solving Wiener–Hopf equations (see e.g., Achenbach [9], Carrier et al. [38], and Ablowitz and Fokas [43]). The function (D/a_+) is analytic in the ζ-plane cut along the interval $(m_- < |\mathrm{Re}(\zeta)| < m,\ \mathrm{Im}(\zeta)=0)$ and behaves like $2m_+^2(m^2-m_-^2)\zeta^2$ {with $m_+ = [(1+\varepsilon)/hp]^{1/2}$ and $m_- = (1+\varepsilon)^{-1/2}$} as $|\zeta|\to\infty$. Consequently, an auxiliary function $S(\zeta)$ is introduced through the definition

$$S \equiv \frac{(D/a_+)}{2m_+^2(m^2-m_-^2)(\zeta^2-\zeta_R^2)}, \tag{32}$$

which possesses the desired asymptotic property $S(\zeta)\to 1$ as $|\zeta|\to\infty$ and, additionally, has neither zeros nor poles in the ζ-plane. The only singularities of S are the branch points $\zeta = \pm m_-$ and $\zeta = \pm m$ [which are shared with the original function (D/a_+)], so it is single-valued in the ζ-plane cut along the interval $(m_- < |\mathrm{Re}(\zeta)| < m,\ \mathrm{Im}(\zeta)=0)$. Then, the standard technique of factorization through the use of Cauchy's integral theorem (see e.g., Achenbach [9], Carrier et al. [38], and Ablowitz and Fokas [43]) allows writing

$$S = S^+ \cdot S^-, \tag{33}$$

where S^+ and S^- are analytic functions in the overlapping half-planes $\mathrm{Re}(\zeta) > -m_-$ and $\mathrm{Re}(\zeta) < m_-$, respectively. These are given by

$$S^{\pm}(\zeta) = \exp\left[-\frac{1}{\pi}\int_{m_-}^{m} \arctan\left(\frac{4\omega^2|a_-||\beta|}{T^2}\right)\frac{d\omega}{\omega\pm\zeta}\right], \tag{34}$$

where

$$a_- \equiv a_-(\omega) = (m_-^2-\omega^2)^{1/2}, \tag{35a}$$

$$\beta \equiv \beta(\omega) = (m^2-\omega^2)^{1/2}. \tag{35b}$$

Further, one may observe from (34) that $S^+(\zeta=0) = S^-(\zeta=0)$ and, therefore, $S(\zeta=0) = [S^+(\zeta=0)]^2$. Now, we exploit the latter observation by taking $\zeta=0$ and also take into account (32)–(34) to obtain the following *explicit* formula for the root of the function (D/a_+). This root defines the speed of thermoelastic Rayleigh waves

$$\zeta_R = \frac{m^2}{[2(m^2-m_-^2)]^{1/2}\cdot S^+(\zeta=0)}. \tag{36}$$

It is noticed, finally, that the inequality $m < \zeta_R$ always holds.

5 Inversion Procedure and Solution in the Physical Space/Time

In view of the definition (11b), one can write the unilateral Laplace transformed vertical displacement in the form

$$U_z(x,y,z=-H,p) = \left(\frac{p}{2\pi i}\right)^2 \int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} U_z^*(q,w,z=-H,p) \cdot e^{pqx}e^{pwy}\,dq\,dw, \tag{37}$$

where U_z^* is given in (22). Next, axisymmetry (circular symmetry) of the problem will become clear and be exploited. To this end, we set $q=i\sigma$ and $w=i\tau$ so that $\zeta^2 \equiv q^2+w^2 = -(\sigma^2+\tau^2) = -\rho^2$, and further consider the polar coordinates (r,ϑ) and (ρ,ϕ) defined through the relations $x+iy = re^{i\vartheta}$ and $\sigma+i\tau = \rho e^{i\phi}$. The first set of polar coordinates refers to the physical plane (x,y), whereas the second set to the transform plane (σ,τ). Considering also the case $x\geq 0$ and $y\geq 0$ (which, as will become clear soon, does not impose any restriction to the solution), it should be $\mathrm{Im}(\sigma)\geq 0$ and $\mathrm{Im}(\tau)\geq 0$, whereas $\rho \equiv (\sigma^2+\tau^2)^{1/2} \geq 0$.

Now, in view of (22), (31) and the newly introduced polar coordinates, we obtain

$$U_z(r,\theta,z=-H,p) = \frac{\kappa Q V_1}{4\pi^2}\int_0^{\infty}\left(\int_0^{2\pi}\frac{a_+Te^{-a_-Hp} - a_-Te^{-a_+Hp}}{D(\rho,p)} \times \exp(-ipr\rho\cdot\cos(\phi-\vartheta))\,d\phi\right)\rho\,d\rho, \tag{38}$$

and further, by observing that the inner integral in (38) is actually independent on the starting limit of the integration interval, we eliminate the variable ϑ from the problem and get

$$U_z(r,\theta,z=-H,p)$$
$$=\frac{\kappa QV_1}{4\pi^2}\int_0^\infty\left(\frac{a_+Te^{-a_-Hp}-a_-Te^{-a_+Hp}}{D(\rho,p)}\times\int_0^{2\pi}\exp(-ipr\rho\cdot\cos\phi)d\phi\right)\rho d\rho$$
$$=\frac{\kappa QV_1}{2\pi}\int_0^\infty\frac{a_+Te^{-a_-Hp}-a_-Te^{-a_+Hp}}{D(\rho,p)}J_0(pr\rho)\cdot\rho d\rho. \tag{39}$$

In the above relation, we have $a_\pm=(m_\pm^2+\rho^2)^{1/2}$, $\beta=(m^2+\rho^2)^{1/2}$, $T=m^2+2\rho^2$, and

$$D(\rho,p)=-\frac{1+\varepsilon}{hp}\left(\frac{1+\varepsilon}{hp}+\rho^2\right)^{1/2}\left[(m^2+2\rho^2)^2-4\rho^2\left(\frac{1}{1+\varepsilon}+\rho^2\right)^{1/2}(m^2+\rho^2)^{1/2}\right]=-\frac{1+\varepsilon}{hp}\left(\frac{1+\varepsilon}{hp}+\rho^2\right)^{1/2}R^{\text{therm}}, \tag{40}$$

while the following standard result for the Bessel function $J_0(\,)$ was used (see e.g., Watson [44])

$$\frac{1}{2\pi}\int_0^{2\pi}\exp(-ipr\rho\cdot\cos\phi)d\phi=J_0(pr\rho). \tag{41}$$

One may observe that the last integral in (39) is but an *inverse* Hankel transform (see e.g., Bracewell [45]). This confirms the circular symmetry of the problem.

Next, another change of variable defined by setting $\omega=pr\rho$ leads to the following expression for a normalized expression of the unilateral Laplace-transformed vertical displacement at the surface

$$U_z^{\text{norm}}(r,z=-H,p)=-\frac{1}{2(m^2-m_-^2)r^2}\times\int_0^\infty\frac{\left(e^{-a_-Hp}-\frac{a_-}{a_+}e^{-a_+Hp}\right)T}{m_+^2\left(\frac{\omega^2}{p^2r^2}+\zeta_R^2\right)\cdot S\left(i\frac{\omega}{pr}\right)}\times\frac{\omega}{p^2}J_0(\omega)d\omega, \tag{42}$$

where $U_z^{\text{norm}}=(2\pi U_z)/(\kappa QV_1)$, and the symbols a_+, a_-, and T take the following forms [which, of course, follow from the definitions in (14*a*) and (17*a*) and the several changes of variable in the previous analysis]

$$a_\pm=\left(m_\pm^2+\frac{\omega^2}{p^2r^2}\right)^{1/2},\qquad T=m^2+2\frac{\omega^2}{p^2r^2}. \tag{43a,b}$$

Finally, we consider the inverse unilateral Laplace transform in (10*b*) and, further, interchange the latter integration and the integration in (42). This is permissible since the integral in (10*b*) converges *uniformly* within its region of convergence in the complex p-plane [37]. Cf. Miklowitz [36] and Markenscoff and Ni [46], e.g., for similar interchanges of the integration order in multiple transform inversions. In this way, we obtain the normalized vertical displacement at the surface, $u_z^{\text{norm}}=(2\pi u_z)/(\kappa QV_1)$, in the physical space/time

$$u_z^{\text{norm}}(r,z=-H,s)=-\frac{1}{4\pi i}\frac{1}{(m^2-m_-^2)\zeta_R^2r^2}\cdot\int_0^\infty\left(\int_{c-i\infty}^{c+i\infty}\frac{\left(e^{-a_-Hp}-\frac{a_-}{a_+}e^{-a_+Hp}\right)T}{m_+^2(p-ip_R)(p+ip_R)\cdot S\left(i\frac{\omega}{pr}\right)}e^{ps}dp\right)\times\omega J_0(\omega)d\omega, \tag{44}$$

where

$$a_+=\left(\frac{1+\varepsilon}{h}\right)^{1/2}\frac{1}{p}\gamma_+, \tag{45a}$$

$$a_-=\frac{1}{(1+\varepsilon)^{1/2}p}\gamma_-, \tag{45b}$$

$$\gamma_+\equiv\gamma_+(p,\omega)=\left(p+\frac{h}{(1+\varepsilon)r^2}\omega^2\right)^{1/2}, \tag{46a}$$

$$\gamma_-\equiv\gamma_-(p,\omega)=\left(p^2+\frac{(1+\varepsilon)}{r^2}\omega^2\right)^{1/2}, \tag{46b}$$

and

$$p_R=\frac{\omega}{\zeta_Rr}. \tag{47}$$

Notice that the branch cuts depicted in Fig. 3 should be introduced to render the functions γ_+ and γ_- single-valued. Also, the constant c in the inner integral of (44) is taken slightly greater than zero since all *singularities* (poles and branch points) of the corresponding integrand are situated in the plane $\text{Re}(p)\leqslant 0$. Specifically, these singularities include the poles at $\pm ip_R$, the branch point $-h\omega^2((1+\varepsilon)r^2)^{-1}$ for the function γ_+, and the branch points $\pm i\omega(m_-r)^{-1}$ for the function γ_- (see Fig. 3).

With the above results available, we now focus interest on the thermoelastic Rayleigh waves. As is well-known (see e.g., Chao et al. [47], Achenbach [9], and Miklowitz [36]), analytically the Rayleigh-wave effects correspond to the contributions from certain poles in the integrands of the inversion integrals. Indeed, in our previous analysis we were able to make explicit the appearance of Rayleigh-wave poles [cf. Eq. (44)].

We proceed now to evaluate the pole contribution in (44) obtaining therefore an approximate solution for the thermoelastic Rayleigh-wave signals along the half-space surface. Care should be exercised, however, in evaluating the functions $a_+(p,\omega)$ and $a_-(p,\omega)$ at the points $\pm ip_R$, which lie along the Bromwich path $(c-i\infty,c+i\infty)$ (see Fig. 3). The following results are obtained:

$$a_+=\mp i\zeta_R\ell_R^{1/2}\exp(\pm i\theta/2), \tag{48a}$$

$$a_-=\mp i|a_{R-}|\quad\text{at}\quad p=\pm ip_R, \tag{48b}$$

where

$$a_{R-}=(m_-^2-\zeta_R^2)^{1/2}, \tag{49a}$$

$$\ell_R=(1+\tan^2\theta)^{1/2}, \tag{49b}$$

$$\tan\theta=\frac{(1+\varepsilon)r}{h\zeta_R\omega}. \tag{49c}$$

The symbol θ in the above relations (denoting an angle in Fig. 3) should not be confused with the symbol used earlier to denote the change in temperature. Then, (44) provides the following expression for the disturbance due to the thermoelastic Rayleigh waves

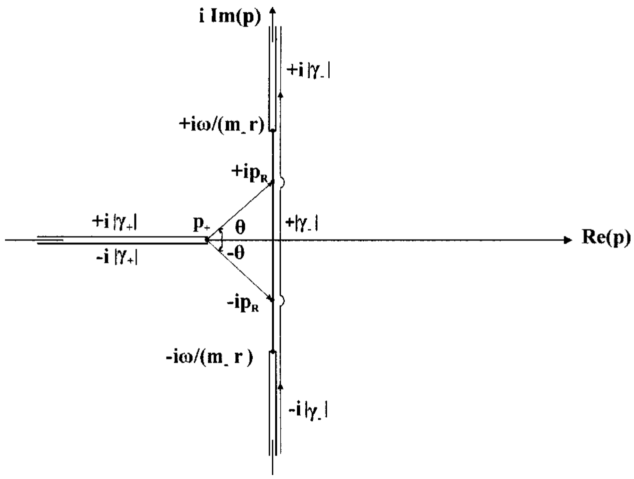

Fig. 3 Branch cuts for the functions $\gamma_{+}(p,\omega)$ and $\gamma_{-}(p,\omega)$, and the Bromwich path in the complex p-plane

$$u_z^{\text{norm}}(r,z=-H,s)=-\frac{\Lambda}{r^2}\int_0^{\infty} f(r,H,\omega,s)\cdot\omega\cdot J_0(\omega)\,d\omega, \tag{50}$$

where

$$\begin{aligned} f(r,H,\omega,s)=&\exp\left(-\frac{\omega|a_{R-}|H}{\zeta_R r}\right)\cos\left(\frac{\omega s}{\zeta_R r}\right)\\ &-\frac{|a_{R-}|\exp\left(-\dfrac{\omega}{r}\ell_R^{1/2}H\cos(\theta/2)\right)}{\zeta_R\ell_R^{1/2}}\\ &\times\cos\left[\frac{\omega s}{\zeta_R r}-\frac{\theta}{2}-\left(\frac{\omega}{r}\ell_R^{1/2}H\sin(\theta/2)\right)\right], \end{aligned} \tag{51}$$

$$\Lambda=\frac{m^2-2\zeta_R^2}{4\left(\dfrac{1+\varepsilon}{h}\right)\cdot S(\zeta_R)\cdot(m^2-m_-^2)\zeta_R^2}. \tag{52}$$

Further, as the analysis in Appendix B shows, the second term of $f(r,H,\omega,s)$ in (51) is negligible with respect to the first term. Omitting the small term, the normalized vertical displacement becomes

$$\begin{aligned} u_z^{\text{norm}}(r,z=-H,s)=&-\frac{\Lambda}{r^2}\int_0^{\infty}\exp\left(-\frac{\omega|a_{R-}|H}{\zeta_R r}\right)\\ &\times\cos\left(\frac{\omega s}{\zeta_R r}\right)J_0(\omega)\,\omega\,d\omega. \end{aligned} \tag{53}$$

Finally, evaluation of the integral in (53) (Watson [44]) yields the following *closed-form* expression for the normalized vertical displacement at the surface due to Rayleigh waves generated by a buried thermal source in a half-space

$$u_z^{\text{norm}}(r,z=-H,s)=-\frac{\Lambda}{r^2}\,\text{Re}\left[\frac{\dfrac{|a_{R-}|H}{\zeta_R r}-i\dfrac{s}{\zeta_R r}}{\left[1+\left(\dfrac{|a_{R-}|H}{\zeta_R r}-i\dfrac{s}{\zeta_R r}\right)^2\right]^{3/2}}\right], \tag{54}$$

where r is the radial distance from the epicenter, Re[ ] denotes the real part of a complex function, and the quantities Λ, a_{R-} and ζ_R depend on the material constants. It is also of notice that u_z^{norm} depends on the ratio (H/r).

6 Numerical Results, Further Asymptotic Results, and Concluding Remarks

Numerical results from the previous expression are obtained easily through the use of MATHEMATICA™ for both numerical integrations and symbolic manipulations involved. A model material was considered to derive the results shown in the graphs of Figs. 4 and 5. It is characterized by the following constants: coupling constant $\varepsilon=0.011$, thermoelastic characteristic length $h=1.864\times10^{-9}$ m, and Poisson's ratio $\nu=0.3$ [this value gives a ratio of wave velocities $m\equiv(V_1/V_2)=1.8708$]. The graphs present the variation of the normalized vertical displacement $u_z^{\text{norm}}\equiv 2\pi u_z(\kappa QV_1)^{-1}$ with the normalized time $V_1(r^2+H^2)^{-1/2}t$. In Fig. 4, the case $r=10H$ is considered, whereas in Fig. 5, both cases $r=40H$ and $r=160H$ are presented. In all cases, $H=100$ m is taken but a numerical inspection showed that the shape of pulse does not change appreciably if normalization is utilized (the displacement itself becomes larger for smaller depths).

The graphs show the generation of the thermoelastic Rayleigh wave at the half-space surface. We notice that as the distance of the observation station from the epicenter increases, the shape of the Rayleigh disturbance appears to become sharper because of the contraction of the real time scale with the increase of the length $(r^2+H^2)^{1/2}$. Also, as the observation station moves away from the epicenter, *decay* in amplitude occurs after a certain point. This attenuation is due to the 3D geometry of the problem (see for analogous nonthermal situations in Pekeris and Lifson [11], and Achenbach [9]). On the contrary, the latter result is not encountered in the respective 2D problem of a nonthermal buried dilatational source treated by Garvin [10], where once the Rayleigh pulse takes its shape, it is not decaying.

In the sequel, we further investigate the behavior of $u_z^{\text{norm}}(r,z=-H,s)$ at *large* distances from the epicenter, i.e., for $r\gg H$. In this case, it is $(r^2+H^2)^{1/2}\cong r$ and the normalized vertical displacement takes the form

$$u_z^{\text{norm}}(r\gg H,z=-H,s)\cong-\Lambda\,\text{Re}\left[\frac{X-iYr}{(r^2+X^2-Y^2r^2-i2XYr)^{3/2}}\right], \tag{55}$$

where $X=|a_{R-}|\cdot H/\zeta_R$ and $Y=V_Rt/r$. Taking $r\gg H$ leads to the conclusion that $r\gg X$ and then (55) takes the even simpler form

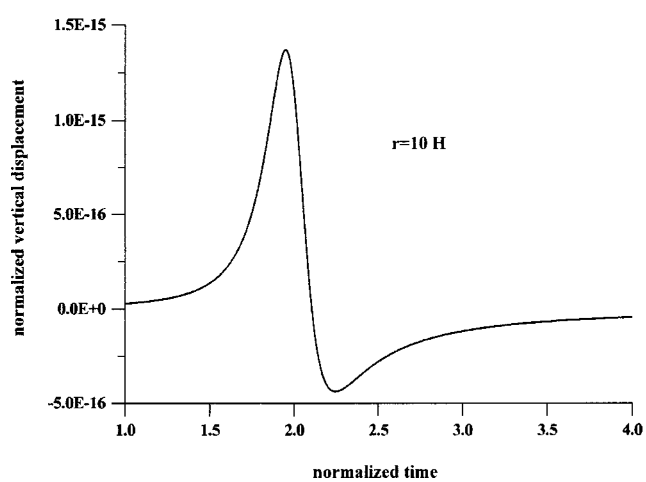

Fig. 4 The variation of the normalized vertical displacement $u_z^{\text{norm}}\equiv 2\pi u_z(\kappa QV_1)^{-1}$ with the normalized time $s(r^2+H^2)^{-1/2}$ indicating the arrival of a thermoelastic Rayleigh wave at the station $r=10H$. The constants have the values $\varepsilon=0.011$, $h=1.864\times10^{-9}$ m, $\nu=0.3$, and $H=100$ m.

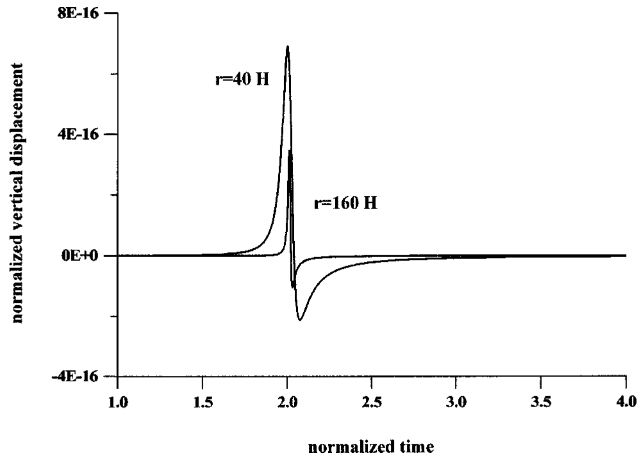

Fig. 5 The variation of the normalized vertical displacement $u_z^{\mathrm{norm}} \equiv 2\pi u_z(\kappa Q V_1)^{-1}$ with the normalized time $s(r^2+H^2)^{-1/2}$ indicating the arrival of a thermoelastic Rayleigh wave at the stations $r=40\,H$ and $r=160\,H$. The constants have the values $\varepsilon=0.011$, $h=1.864\times10^{-9}$ m, $\nu=0.3$, and $H=100$ m.

$$u_z^{\mathrm{norm}}(r \gg H, z=-H, s) \cong -\Lambda\,\mathrm{Re}\left[\frac{X-iYr}{(r^2-Y^2r^2-i2XYr)^{3/2}}\right]. \tag{56}$$

In addition, we can investigate the field *near* the Rayleigh wavefront. To this end, the transformation $x_R=V_Rt-r$ is used in (56), where x_R denotes the distance from the Rayleigh wavefront, providing

$$u_z^{\mathrm{norm}}(r \gg H, z=-H, s) \cong -\frac{\Lambda}{r^{1/2}}\mathrm{Re}\left[\frac{\dfrac{X}{r}-i\left(\dfrac{x_R}{r}+1\right)}{\left[-x_R\left(\dfrac{x_R}{r}+2\right)-i2X\left(\dfrac{x_R}{r}+1\right)\right]^{3/2}}\right]. \tag{57}$$

Now, by taking $r \gg X$ and $r \gg x_R$, we get the following expression for the vertical displacement far from the epicenter and, at the same time, very close to the Rayleigh wavefront

$$u_z^{\mathrm{norm}}(r \gg H, r \gg x_R, z=-H, s) \cong -\frac{\Lambda}{(8r)^{1/2}}\mathrm{Re}\left[\frac{-i}{(-x_R-iX)^{3/2}}\right]. \tag{58}$$

The above relation reveals that the displacement varies with the distance from the epicenter as $u_z^{\mathrm{norm}}(r \gg H, r \gg x_R, z=-H, s) \approx r^{-1/2}$, while in the case of a source that is situated very close to the surface (i.e., when $x_R \gg X$) the displacement varies with the distance from the Rayleigh wavefront as $u_z^{\mathrm{norm}}(r \gg H, r \gg x_R, z=-H, s) \approx x_R^{-3/2}$. The first of the aforementioned results shows that the surface effects attenuate with distance as $r^{-1/2}$, the physical explanation of which is that the surface waves in our 3D problem are essentially cylindrical waves (see Ref. [9] for analogous situations in classical elastodynamics).

In conclusion, the 3D transient dynamic problem of a thermoelastic half-space under thermal buried or surface loading is treated in this paper. The loading has the form of a concentrated heat flux applied impulsively and Biot's fully coupled thermoelasticity is utilized. The problem has relevance to situations involving heat generation due to, e.g., laser action (impulsive electromagnetic radiation) on a surface target, underground nuclear activity, and friction developed during underground fault motions. Here, we were particularly interested in determining in *closed form* the disturbance associated with the propagation of the thermoelastic Rayleigh waves. This was made possible by using multiple Laplace transforms, asymptotics, complex-variable theory and certain results for Bessel functions. The dependence of the normal displacement associated with the Rayleigh wave upon the distance from the source epicenter and the distance from the wavefront was also determined.

Appendix A

The solution of the 12×12 algebraic system of Sec. 3 reads

$$X_1=-\kappa QV_1e^{-2a_-Hp}\Bigg(-(e^{2a_-Hp}-e^{-2(a_+-a_-)Hp})a_-M_-TE + \left(e^{2a_-Hp}+e^{-2(a_+-a_-)Hp}-2\frac{M_-}{M_+}e^{-(a_+-a_-)Hp}\right)a_+M_+TE + (e^{2a_-Hp}+e^{-2(a_+-a_-)Hp})D\Bigg)\Big/(BA), \tag{A1}$$

$$X_2=\kappa QV_1\Bigg(-a_-e^{-(a_++3a_-)Hp}\Bigg[-2e^{2a_-Hp}+(e^{(a_++a_-)Hp} + e^{(a_++3a_-)Hp})\frac{M_-}{M_+}\Bigg]\cdot M_+TE+e^{-2a_-Hp}((-1 + e^{2a_-Hp})M_+a_+TE+(1+e^{2a_-Hp})D)\Bigg)\Big/(CA), \tag{A2}$$

$$X_3=-2\kappa QV_1\beta qTe^{-(a_++a_-+\beta)Hp}(a_+e^{a_+Hp}-a_-e^{a_-Hp})(T_q + w^2)/(FA), \tag{A3}$$

$$X_4=2\kappa QV_1\beta wTe^{-(a_++a_-+\beta)Hp}(a_+e^{a_+Hp}-a_-e^{a_-Hp})(T_w + q^2)/(FA), \tag{A4}$$

$$X_5=-\frac{\kappa QV_1}{2a_+m^2p(M_+-M_-)}=-\frac{\kappa QV_1}{B}, \tag{A5}$$

$$X_6=-\kappa QV_1e^{-(2a_++a_-)Hp}\{a_-e^{a_-Hp}M_-TE+a_+((e^{a_-Hp}M_+ - 2e^{a_+Hp}M_-)TE)+e^{a_-Hp}D/(BA)\}, \tag{A6}$$

$$X_7=\frac{\kappa QV_1}{C}, \tag{A7}$$

$$X_8=\kappa QV_1e^{-(a_++2a_-)Hp}(a_-(2e^{a_-Hp}M_+-e^{a_+Hp}M_-)TE + a_+e^{a_+Hp}(M_+T(-E))+e^{a_+Hp}D)/(CA), \tag{A8}$$

$$X_9=X_{10}=0, \tag{A9}$$

$$X_{11}=-2\kappa QV_1\beta qTe^{-(a_++\beta)Hp}(-a_-+a_+e^{(a_+-a_-)Hp})(T_q + w^2)/(FA), \tag{A10}$$

$$X_{12}=2\kappa QV_1\beta wTe^{-(a_++a_-+\beta)Hp}(-a_+e^{a_+Hp}+a_-e^{a_-Hp})(T_w + q^2)/(FA), \tag{A11}$$

where

$$A=(a_+M_+-a_-M_-)T(T_qT_w-q^2w^2)+4a_+a_-\beta(M_+-M_-) \times(q^2T_q+2q^2w^2+w^2T_w), \tag{A12}$$

$$B=2a_+m^2(M_+-M_-)p, \tag{A13}$$

$$C=2a_-m^2(M_+-M_-)p, \tag{A14}$$

$$D=4a_+a_-\beta(M_+-M_-)(q^2T_q+2q^2w^2+w^2T_w), \tag{A15}$$

$$E=T_qT_w-q^2w^2, \tag{A16}$$

$$F=m^2p. \tag{A17}$$

Appendix B

Considering the ratio $\Re$ of the fluctuation amplitudes of the first and second terms in (51), we obtain

$$\Re=\frac{\zeta_R l_R^{1/2}}{|a_{R-}|}\exp\left(\frac{\omega}{r}H\left[l_R^{1/2}\cos(\theta/2)-\frac{|a_{R-}|}{\zeta_R}\right]\right). \qquad (B1)$$

We will examine this ratio and conclude that $\Re$ takes on very large values in the entire range of ω provided that the distance from the epicenter r is much greater than the thermoelastic characteristic length h. The length h is *very small* for most materials $[h=O(10^{-10}\,\mathrm{m})$ as mentioned in the main text of the paper$]$ and, therefore, the requirement $(r/h)\gg 1$ does not pose any serious limitation. Similarly to the case of Eq. (21*b*), any choice with, say, $(r/h)\geqslant 100$ leads to a reasonable approximation. To cover the entire range of ω-values, we discern the following possibilities:

(1) Considering $(\omega/r)\to 0$, Eqs. (49*c*) and (49*b*) provide the results $\lim_{(\omega/r)\to 0}\tan\theta=\infty$, $\lim_{(\omega/r)\to 0}\ell_R^{1/2}=\infty$ and $\lim_{(\omega/r)\to 0}\cos(\theta/2)=2^{-1/2}$. Then, we find that $\lim_{(\omega/r)\to 0}((\omega/r)H[l_R^{1/2}\cos(\theta/2)-(|a_{R-}|/\zeta_R)])=0$ and $\lim_{(\omega/r)\to 0}\Re=\infty$.

(2) Considering $(\omega/r)\to\infty$, Eqs. (49*c*) and (49*b*) provide the results $\lim_{(\omega/r)\to\infty}\tan\theta=0$, $\lim_{(\omega/r)\to\infty}\ell_R^{1/2}=1$ and $\lim_{(\omega/r)\to\infty}\cos(\theta/2)=1$. Also, from (36) and (49*a*) we may infer that $(|a_{R-}|/\zeta_R)<1$. Then, working as in the above case, we find that $\lim_{(\omega/r)\to\infty}\Re=\infty$.

(3) Considering $\omega=O(r)$, Eqs. (49*c*) and (49*b*) lead us to conclude that $\tan\theta$ and $\ell_R^{1/2}$ takes on very large values. Consequently, it is $\cos(\theta/2)\cong 2^{-1/2}$ and by (*B*1) it is seen that $\Re$ takes on very large values in this case too.

Finally, we note that a numerical evaluation of the two terms in (51) showed that the amplitude of the first term is, at minimum, about 20 orders of magnitude greater than the amplitude of the second term.

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